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A rotating spherical liquid drop in an electric field

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It is shown that the equilibrium shape of an incompressible dielectric fluid drop rotating with constant angular velocity in the presence of a uniform external electric field of appropriate magnitude along the axis of rotation is spherical. For an inviscid fluid drop, the stability of this spherical configuration to small deformations is investigated by means of Chandrasekhar's virial method. We find that a rotating drop in the presence of an electric field parallel to the axis of rotation is, in some respects, more stable than when either only the electric field or only rotation is present. This is due to the fact that the application of an electric field parallel to the axis of a rotating drop, or of rotation parallel to an electric field in which a drop is immersed, shifts the instability mechanism to another normal mode.

1. Introduction

The equilibrium configuration and the stability conditions of a fluid drop have received considerable attention in the literature. Particular attention has been paid to a rotating drop and to the mechanics of disintegration of a drop by electrostatic forces. Rayleigh (1882) showed that a conducting spherical drop of radius a carrying a charge Q becomes unstable when $Q^2 > 16\pi a^3 T$, where T is the surface tension. The case of an uncharged conducting fluid drop in an otherwise uniform external field was first investigated experimentally by Zeleny (1917). The drop becomes elongated in the direction of the field, taking an approximately spheroidal shape, and eventually, that is when the field is sufficiently strong, it bursts. Zeleny's (1915) theoretical analysis, however, is incorrect, as was pointed out by Taylor (1964). Indeed, Taylor (1964) re-examined this problem assuming that the drop is approximately spheroidal. In this case the stress of the electric field cannot balance the stress due to surface tension everywhere on the spheroid. Taylor used an approximate equilibrium configuration by requiring these stresses and the constant hydrostatic pressure to balance only at the equator and the poles of the spheroid. Recently Brazier-Smith (1971) showed that there are finite shape-preserving oscillations, obeying the Taylor approximation of a spheroidal drop in an electric field.

The problem of a dielectric fluid drop in an electric field was also considered by Garton & Krasucki (1964) and Rosenkilde (1969). Garton & Krasucki assumed that the electric field in the drop is a constant and solved the exact equation for obtaining the boundary of the drop. Since, however, the drop cannot be exactly spheroidal the assumption of a constant field in the interior of the drop involves an approximation. Rosenkilde also assumed that the drop in an electric field retains a spheroidal shape, and used Chandrasekhar's virial method for determining the equilibrium configuration and its stability to small oscillations. In these studies the fluid in which the drop is suspended is an insulator. When the fluid surrounding the drop is conducting there is an imbalance in the tangential component of the electric field stress over the drop surface and this generates a flow field (Taylor 1966; Torza, Cox & Mason 1971).

The case of a rotating fluid drop has been considered by many authors, including Chandrasekhar (1965). Brazier-Smith, Jennings & Latham (1972) employed a rotating drop model in connexion with the coalescence of falling water drops. A rotating drop is oblate in the direction of rotation and for relatively low angular velocities it has the shape of an oblate spheroid of small eccentricity. For relatively large angular velocities the shape of the drop departs appreciably from that of a spheroid and eventually the drop becomes unstable.

An account of the effects of electric fields on hydrodynamic stability is given in an interesting paper by Calvert & Melcher (1969), which contains many useful references to related work. Calvert & Melcher were interested in the application of electric fields to problems connected with the cryogenic management of fluids in weightless space conditions. They investigated theoretically and experimentally the stability of a circular cylindrical column of fluid having solid-body rotation about its axis, in the presence of an axial electric field which has a large radial gradient at the interface. Their work showed that a sufficiently strong electric field makes the purely azimuthal waves completely stable.

In this paper we consider the problem of a rotating drop in the presence of an electric field parallel to the axis of rotation. In order to simplify the analysis we assume that the fluid in which the drop is suspended is an inviscid insulator exerting a uniform constant pressure over the drop surface. We show that, when there is a suitable relationship between the angular velocity and the electric field, an incompressible drop retains a spherical shape, and all the boundary conditions are satisfied. It might be observed that, since an electric field makes the drop prolate and rotation makes it oblate, a suitable combination of angular velocity and electric field will be expected to leave the drop spherical. This argument, however, is not necessarily correct. For example, it is not correct for a compressible drop. In the case of an incompressible drop, it is a happy coincidence that the normal stress (for a sphere) due to a uniform external field has the same angular dependence as the pressue due to solid-body rotation of the drop. Thus a suitable combination of these two factors leaves the drop spherical. The surface tension affects the stability of the drop, which we also investigate.

2. The equilibrium configuration

We consider an incompressible fluid drop of density ρ rotating with constant angular velocity Ω between parallel electrodes that generate an electric field $\mathbf{F}$. We assume that the axis of rotation is parallel to the undisturbed field $\mathbf{F}$ and that the drop retains a spherical shape of radius a. In a spherical polar co-ordinate system (r, θ, ϕ), with the origin at the centre of the drop and the axis $\theta = 0$ along the axis of rotation, the electric field is given by

$$\mathbf{F} = \begin{cases} F\left[\left(1-2\dfrac{\epsilon_1-\epsilon_2}{2\epsilon_1+\epsilon_2}\dfrac{a^3}{r^3}\right)\cos\theta, \quad -\left(1+\dfrac{\epsilon_1-\epsilon_2}{2\epsilon_1+\epsilon_2}\dfrac{a^3}{r^3}\right)\sin\theta, 0\right], \quad (r > a), \\ \dfrac{3\epsilon_1 F}{2\epsilon_1+\epsilon_2}(\cos\theta, \ -\sin\theta, 0), \quad (r < a), \end{cases} \tag{1}$$

where ϵ_1 and ϵ_2 are the permittivities of the external medium and the drop, respectively. When $\epsilon_2 = \infty$ we have the case of conducting drop. The electric field stress normal to the surface of the drop in the outward direction is

$$\frac{1}{4\pi}(-\tfrac{1}{2}\epsilon_1 F_+^2 + \tfrac{1}{2}\epsilon_2 F_-^2 + \epsilon_1 F_{r+}^2 - \epsilon_2 F_{r-}^2) = \frac{9F^2\epsilon_1}{8\pi(2\epsilon_1+\epsilon_2)^2}(\epsilon_2-\epsilon_1)[\epsilon_2-(\epsilon_2-\epsilon_1)\sin^2\theta], \tag{2}$$

where the plus sign refers to the external field, the minus to the internal field and the suffix r to the radial component. The pressure p within the rotating drop is given by

$$p = p_0 + \tfrac{1}{2}\rho\Omega^2 r^2 \sin^2\theta, \tag{3}$$

where p_0 is a positive constant.

At the surface of the drop the normal stresses must balance, that is,

$$p_0 + \tfrac{1}{2}\rho\Omega^2 a^2 \sin^2\theta + \frac{9F^2\epsilon_1}{8\pi(2\epsilon_1+\epsilon_2)^2}(\epsilon_2-\epsilon_1)[\epsilon_2-(\epsilon_2-\epsilon_1)\sin^2\theta] = \frac{2T}{a} + p_1, \tag{4}$$

where p_1 is the hydrostatic external pressure. If in (4) we equate to zero the coefficient of $\sin^2\theta$ we obtain

$$\rho a^2 \Omega^2 = 9\epsilon_1(\epsilon_2-\epsilon_1)^2 F^2/4\pi(2\epsilon_1+\epsilon_2)^2, \tag{5}$$

which is the condition that the drop remains spherical. When (5) is not quite satisfied the drop will become prolate or oblate. Since $\mathbf{F}$ elongates the drop in a direction parallel to itself and rotation compresses the drop along the axis of rotation, the drop will become prolate or oblate depending on whether the right-hand side of (5) is greater or smaller than its left-hand side.

Since p_0 must not be negative, (4) and (5) show that if $p_1 = 0$ then

$$T > 9a\epsilon_1\epsilon_2(\epsilon_2-\epsilon_1)F^2/16\pi(2\epsilon_1+\epsilon_2)^2. \tag{6}$$

As was pointed out by Taylor (1964), there is no reason why p_1 should be zero, and the above inequality is an unnecessary restriction, thus we ignore it. Then (5) shows that for a given dielectric liquid drop and a given electric field there is always an angular velocity that will make the drop spherical irrespective of the surface tension. The surface tension affects the stability of the drop, which is discussed in the following section.

3. The perturbation equations

We are going to consider the stability of small oscillations of the system by means of the virial method developed by Chandrasekhar (see, for example, the various articles by Chandrasekhar and Lebovitz that appeared in the

Astrophysical Journal in the 1960's). It is thus convenient to introduce, in this section, Cartesian co-ordinates (x_1, x_2, x_3), where $x_1 = r\sin\theta\cos\phi, x_2 = r\sin\theta\sin\phi$ and $x_3 = r\cos\theta$. We neglect viscosity and consider a Lagrangian perturbation of the steady state of the form $\boldsymbol{\xi}(\mathbf{x})\, e^{\lambda t}$, where λ is a characteristic value which is to be determined.

We define

$$V_{i;j} = \int_V \rho \xi_i x_j \, dV \quad (i = 1, 2, 3), \tag{7}$$

where the integration is carried throughout the volume V of the drop, and $V_{i;j}$ is related to the variation of the moment of inertia tensor by

$$\delta \int \rho x_i x_j \, dV = V_{i;j} + V_{j;i} = V_{ij}. \tag{8}$$

In this work the summation convention for repeated indices does not apply except for the index l.

Following Chandrasekhar we assume that

$$\xi_i = a_{il} x_l, \tag{9}$$

where the a_{il} are constants and therefore

$$V_{i;j} = \tfrac{4}{15}\pi\rho a^5 a_{ij}.$$

Since for an incompressible fluid $\nabla.\boldsymbol{\xi} = 0$, we must have

$$V_{11} + V_{22} + V_{33} = 0. \tag{10}$$

In a frame of reference rotating with the fluid angular velocity Ω, the variation of the equilibrium virial equations of our problem to the first order in $\boldsymbol{\xi}$ is given by

$$\lambda^2 V_{i;j} - 2\lambda\Omega\epsilon_{il3} V_{l;j} - \Omega^2 (V_{ij} - \delta_{i3} V_{3j}) = \delta B_{ij} + \delta G_{ij} + \delta S_{ij} + \delta_{ij}\delta\Pi. \tag{11}$$

In (11), B_{ij} is the stress tensor due to the electric field. For an ellipsoid, and thus for a sphere, its variation when ξ_i is given by (9) was calculated by Rosenkilde. G_{ij} is the normal stress due to the surface tension and its variation was calculated by Chandrasekhar.

$$\delta S_{ij} = -\delta \int_S p_1 x_j \, dS_i, \quad \delta\Pi = \delta \int_V p \, dV,$$

the surface integral being taken over the surface of the drop. It can easily be shown that if p_1 is a constant, that is if we neglect the effect of the perturbation on the pressure exterior to the fluid drop,

$$\delta S_{ij} = 0 \quad \text{when} \quad \nabla.\boldsymbol{\xi} = 0.$$

On making use of (10), after some algebra, we find that the nine equations represented by (9) are

$$\lambda^2 V_{1;2} - 2\lambda\Omega V_{2;2} = (B + 5C)(V_{1;2} + V_{2;1}), \tag{12}$$

$$\lambda^2 V_{2;1} + 2\lambda\Omega V_{1;1} = (B + 5C)(V_{1;2} + V_{2;1}), \tag{13}$$

$$\lambda^2 V_{1;1} - 2\lambda\Omega V_{2;1} = \delta\Pi + 2BV_{1;1} + C\left[\frac{42(4\epsilon_1 + \epsilon_2)}{2\epsilon_1 + \epsilon_2} V_{3;3} + 15V_{1;1} + 5V_{2;2} - 20V_{3;3}\right], \tag{14}$$

$$\lambda^2 V_{2;2} + 2\lambda\Omega V_{1;2} = \delta\Pi + 2BV_{2;2} + C\left[\frac{42(4\epsilon_1+\epsilon_2)}{2\epsilon_1+\epsilon_2}V_{3;3} + 5V_{1;1} + 15V_{2;2} - 20V_{3;3}\right], \quad (15)$$

$$\lambda^2 V_{3;3} = \delta\Pi - \left(\frac{8T}{a^3\rho} + C\frac{6\epsilon_1 - 81\epsilon_2}{2\epsilon_1+\epsilon_2}\right)V_{3;3}, \quad (16)$$

$$\lambda^2 V_{1;3} - 2\lambda\Omega V_{2;3} = \left(B - C\frac{61\epsilon_1-\epsilon_2}{2\epsilon_1+\epsilon_2}\right)(V_{1;3}+V_{3;1}), \quad (17)$$

$$\lambda^2 V_{2;3} + 2\lambda\Omega V_{1;3} = \left(B - C\frac{61\epsilon_1-\epsilon_2}{2\epsilon_1+\epsilon_2}\right)(V_{2;3}+V_{3;2}), \quad (18)$$

$$\lambda^2 V_{3;1} = \left(9C\frac{\epsilon_1+4\epsilon_2}{2\epsilon_1+\epsilon_2} - \frac{4T}{a^3\rho}\right)(V_{3;1}+V_{1;3}), \quad (19)$$

$$\lambda^2 V_{3;2} = \left(9C\frac{\epsilon_1+4\epsilon_2}{2\epsilon_1+\epsilon_2} - \frac{4T}{a^3\rho}\right)(V_{2;3}+V_{3;2}), \quad (20)$$

where $B = \Omega^2 - 4T/a^3\rho$ and $C = 9\epsilon_1(\epsilon_1-\epsilon_2)^2 F^2/140\pi a^2\rho(2\epsilon_1+\epsilon_2)^2$. As in the case where $F = 0$, equations (12)–(20) separate into three non-combining groups, so that for each group only certain virials do not vanish. The normal modes associated with these groups are known as the toroidal, pulsation and transverse-shear modes.

The toroidal modes

For these modes

$$\xi_1 = \alpha x_1 + \beta x_2, \quad \xi_2 = \beta x_1 - \alpha x_2, \quad \xi_3 = 0$$

and thus

$$V_{11} + V_{22} = 0, \quad V_{1;2} = V_{2;1},$$

and all other virials vanish. From (16) it follows that for these modes $\delta\Pi = 0$. On using the above relationships and (5), after a little algebra (12)–(15) give

$$\left\{\left[\lambda^2 - 8\left(\frac{2\Omega^2}{7} - \frac{T}{a^3\rho}\right)\right]^2 + 4\lambda^2\Omega^2\right\}V_{i;j} = 0 \quad (i = 1,2; j = 1,2). \quad (21)$$

If we set $\lambda^2 = -\omega^2$, so that ω real corresponds to stable oscillations, we find that

$$\omega = \pm\left[\Omega \pm \left(\frac{8T}{a^3\rho} - \frac{9\Omega^2}{7}\right)^{\frac{1}{2}}\right]. \quad (22)$$

Thus these modes are stable for $56T > 9a^3\rho\Omega^2$. Equation (21) shows that there is a neutral mode ($\omega = 0$) when $T = \frac{2}{7}a^3\rho\Omega^2$, but since ω is real in the vicinity of this point we do not have instability here. In this case the deformation is time independent and the drop is infinitesimally deformed into an ellipsoid, that is, we have secular instability (Lebovitz 1961).

When $\Omega = 0$, the solutions coalesce to the well-known frequency

$$\omega_0 = (8T/a^3\rho)^{\frac{1}{2}}.$$

The pulsation modes

For these modes

$$\xi_1 = \alpha x_1 + \beta x_2, \quad \xi_2 = -\beta x_1 + \alpha x_2, \quad \xi_3 = -2\alpha x_3$$

and thus

$$V_{1;1} = V_{2;2} = -\tfrac{1}{2}V_{3;3}, \quad V_{1;2} = -V_{2;1},$$

and all the other virials are zero. For this oscillation (12) and (13) become identical and give

$$\lambda^2 V_{1;2} = 2\lambda\Omega V_{1;1}. \tag{23}$$

Equations (14) and (15) also become identical. By eliminating $\delta\Pi$ between one of these and (16) and making use of (5), after some algebra, we obtain

$$\left[3\lambda^2 + \frac{24T}{a^3\rho} + \frac{88\epsilon_1 - 208\epsilon_2}{70\epsilon_1 + 35\epsilon_2}\Omega^2\right] V_{1;1} = -2\lambda\Omega V_{1;2}. \tag{24}$$

From (23) and (24) we obtain

$$\left[3\lambda^2 + \frac{24T}{a^3\rho} + \frac{368\epsilon_1 - 68\epsilon_2}{70\epsilon_1 + 35\epsilon_2}\Omega^2\right] V_{1;1} = 0, \tag{25}$$

and therefore

$$\omega^2 = \frac{8T}{a^3\rho} + \frac{4(92\epsilon_1 - 17\epsilon_2)}{105(2\epsilon_1 + \epsilon_2)}\Omega^2. \tag{26}$$

These modes are stable when the right-hand side of (26) is positive, that is they are stable for all Ω if $\epsilon_2 < \frac{92}{17}\epsilon_1$.

From (23) and (24) it follows that when

$$\frac{3T}{a^3\rho} = \frac{-11\epsilon_1 + 26\epsilon_2}{35(2\epsilon_1 + \epsilon_2)}\Omega^2$$

there is a neutral mode ($\omega = 0$). The occurrence of a neutral mode corresponds to the existence of a suitable deformation that leaves the equilibrium configuration invariant (Chandrasekhar 1965).

The transverse-shear modes

For these modes

$$\xi_1 = \alpha x_3, \quad \xi_2 = \beta x_3, \quad \xi_3 = \gamma x_1 + \delta x_2,$$

the virials $V_{1;3}$, $V_{2;3}$, $V_{3;1}$ and $V_{3;2}$ are proportional to α, β, γ and δ respectively and all the other virials are zero. For a non-zero solution the eliminant of α, β, γ and δ from (17)–(20) must be zero. On making use of (5), after some slight rearrangement, this eliminant becomes

$$\begin{vmatrix} A & A-\lambda^2 & 0 & 0 \\ 0 & 0 & A & A-\lambda^2 \\ A-\lambda^2 & A & -2\lambda\Omega & 0 \\ 2\lambda\Omega & 0 & A-\lambda^2 & A \end{vmatrix} = -\lambda^2[\lambda^2(\lambda^2-2A)^2 + 4\Omega^2(\lambda^2-A)^2]$$

and therefore the equation for the characteristic frequencies of these modes is

$$\omega^2[\omega^2(\omega^2+2A)^2 - 4\Omega^2(\omega^2+A)^2] = 0. \tag{27}$$

In the above expressions

$$A = \frac{9}{35}\frac{\epsilon_1 + 4\epsilon_2}{2\epsilon_1 + \epsilon_2}\Omega^2 - \frac{4T}{a^3\rho}. \tag{28}$$

For stability ω must be real, that is, all the roots of

$$f(x) = x^3 + 4(A - \Omega^2)x^2 + 4A(A - 2\Omega^2)x - 4A^2\Omega^2 = 0, \tag{29}$$

where $x = \omega^2$, must be positive. The condition that the roots of (29) are all real reduces to

$$A[(32\Omega - 13A)^2 + 343A^2] < 0,$$

that is, $A < 0$. Also, $f(0) < 0$, and when $A < 0$ the turning points of $f(x)$ occur at $x > 0$. Thus when $A < 0$ the real roots of (29) are all real and positive.

From (19) and (20) [replacing λ by d/dt] it is easy to see that, when $A = 0$, $V_{3;1}$ and $V_{3;2}$ are of the form $a_1 t + a_2$ and thus $A = 0$ gives an unstable oscillation.

4. Discussion

From (22), (26), (28) and the fact that the transverse-shear mode is unstable when $A \geqslant 0$, it follows that the most unstable mode is the transverse-shear mode. In the absence of rotation the equilibrium configuration is approximately spheroidal and instability sets in through the pulsation mode (Rosenkilde 1969). In the present case instability occurs when

$$\frac{4T}{a^3\rho} = \frac{9}{35}\frac{4\epsilon+1}{\epsilon+2}\Omega^2 = \frac{81\,(4\epsilon+1)\,(\epsilon-1)^2}{140\pi a^2\rho(\epsilon+2)^3}\,\epsilon_1 F^2, \tag{30}$$

where $\epsilon = \epsilon_2/\epsilon_1$. For a given ϵ_1 the coefficient of F^2 in (30) has a maximum at $\epsilon = \frac{1}{31}$ and a minimum at $\epsilon = 1$. From $\epsilon = 1$ it increases monotonically with ϵ to its greatest value, at $\epsilon = \infty$, corresponding to a conducting drop. Therefore the minimum value of F for instability corresponds to $\epsilon = \infty$ and is given by

$$F(a\epsilon_1/T)^{\frac{1}{2}} = \tfrac{2}{9}(35\pi)^{\frac{1}{2}} = 2.330. \tag{31}$$

In the absence of rotation the approximately spheroidal conducting drop becomes unstable for smaller values of the parameter $F(a\epsilon_1/T)^{\frac{1}{2}}$, where a is the radius of the original spherical drop. Taylor's (1964) approximation shows that the minimum critical value of this parameter, corresponding to $\epsilon = \infty$, is 1·625. For a spheroidal conductor the virial method (Rosenkilde 1969) shows that only the pulsation mode can be unstable and the minimum critical value of this parameter is 1·603. Our equation (31) shows that an appropriate rotation increases the minimum critical value of this parameter.

Our results refer to the special case of a spherical drop but have a more general interpretation. Since here instability occurs through the transverse-shear mode, and in the absence of rotation through the pulsation mode, it is obvious that rotation transfers the instability mechanism from one mode to another. Thus if, in the case of a stationary drop elongated by an electric field, immediately before the critical value of $Fa^{\frac{1}{2}}\epsilon^{\frac{1}{2}}/T^{\frac{1}{2}}$ is reached we apply a rotation the drop will become less prolate and thus be able to sustain the stress due to a larger F. Another mode may approach a little nearer its instability limit. Thus if we maintain the approximate equilibrium shape of the drop by increasing F and Ω it is likely, as in the present case, that we shall excite the instability of another mode. Rosenkilde (1969), for example, showed that for the approximate equilibrium configuration he considered, the transverse-shear and toroidal modes are completely stable. The pulsation mode becomes also completely stable if $\epsilon < 20{\cdot}801$. In the spherical configuration considered here [see equation (26)], owing to rotation, this mode becomes stable for a smaller ϵ. It is also obvious that for a given $\epsilon \neq 1$ we can find an Ω and an F that will make the transverse-shear and the toroidal modes unstable. The toroidal mode is the mode which becomes unstable, when $F = 0$. Obviously, in the case of a spherical drop the electric field cannot stabilize this mode.

From the data supplied by Chandrasekhar we find that, when $F = 0$, the toroidal mode becomes unstable when

$$\Sigma = \rho\Omega^2\bar{a}_1^3/8T = 0\cdot463, \tag{32}$$

where $\bar{a}_1$ represents the radius of the undisturbed drop. It is obvious that the above argument for the transfer of instability from one mode to another will also apply as regards the effect of the electric field on the rotating drop. One would therefore expect a rotating drop, in the presence of a suitable electric field, to sustain a higher Ω before the occurrence of instability. This is, indeed, the case in the present example, though here we have a fixed relationship, namely equation (5), between Ω and F. From (30) it is easy to see that in the present case the minimum critical value of Σ, corresponding to $\epsilon = \infty$, is

$$\Sigma = \rho\Omega^2a^3/8T = \tfrac{35}{72} = 0\cdot486. \tag{33}$$

As ϵ decreases from infinity to $\frac{8}{3}$ the minimum value of Σ, for producing instability, increases. At $\epsilon = \frac{8}{3}$ instability sets in either through the transverse-shear mode or [see equation (22)] through the toroidal mode. For $\epsilon < \frac{8}{3}$ instability occurs, as in the case $F = 0$, through the toroidal mode but the critical value of Σ is $\frac{7}{9}$.

The problem described here is an idealized one and the precise occurrence of the instabilities predicted by the theory can only be achieved in the conditions of zero gravity mentioned by Calvert & Melcher. If a rotating drop is suspended under gravity in a nearly inviscid fluid of the same density, in the presence of a suitable electric field, it might possibly achieve a very nearly spherical configuration. The stability of the configuration, however, will be affected by the influence of the perturbations on the fluid surrounding the drop and cannot be in quantitative agreement with the theory. We think, however, that owing to the fact that rotation transfers the instability mechanism from one mode to another, a conducting drop in the presence of an electric field will be more stable when it is suitably rotating. This could be tested by modifying some experimental arrangement so that a drop subjected to a d.c. field is given some rotation.

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